Procedural Phasor Noise, supplemental material

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In this supplemental material we provide derivations for the spectrum and spectrum of variance of the phasor sinewave in the bi-lobe case as well as for the spectrum of variance in the two bilobes case.

1 FOURIER TRANSFORM OF A PERIODIZED NOISE

In order to simplify the analysis of the phasor sinewave, we will restrict ourself to the study of periodic instances of this noise. Indeed, any sparse-convolution noise can easily be restricted to periodic function by sampling kernels in a periodic space. This makes the instantaneous phase of the phasor noise periodic, and hence the phasor noise itself periodic. Lets call T_0 the period and define ω_0 as $\frac{1}{T_0}$. The spectral content of the non-periodic noise can be studied by making T_0 tend towards infinity. In one dimension, the periodic noise can be written has :

$$P(x) = \sum_{k=-\infty}^{+\infty} P_T(x + kT_0) = \left(\left(\sum_{k=-\infty}^{+\infty} \delta_{kT_0} \right) \circledast P_T \right)(x)$$

where $\circledast$ is the convolution product and defining P_T as:

$$P_T(x) = B(x)P(x)$$

provided that B verifies the partition of unity property:

$$\sum_{k=-\infty}^{+\infty} B(x + kT_0) = 1.$$

With this reformulation, the Fourier transform $\mathcal{F}_x[]$ of the periodic noise is :

$$\mathcal{F}_x[P(x)]\,(\omega) = \omega_0 \sum_{k=-\infty}^{+\infty} \mathcal{F}_x[B(x)P(x)](k\omega_0)\, \delta_{k\omega_0}(\omega)$$

assuming that $B(.)\ P(.)$ is an integrable function — implying its Fourier transform is well defined. This assumption is verified if B is also integrable. Thus a good candidate for B is an univariate cardinal B-spline B_n, see e.g. https://en.wikipedia.org/wiki/B-spline.

The amplitude of the deltas in $\mathcal{F}_x[P(x)]$ is defined by the spectrum of $\mathcal{F}_x[B(x)P(x)]$ which is derived in the next section. Note that the choice of a specific function B does not impact the final result (see section 4 for the proof when B-spline are used).

In higher dimensions, the summation variable k in previous equations is replaced by a vector.

For the sake of clarity, lets define $\mathcal{F}_B[]$ has :

$$\mathcal{F}_B[f(\mathbf{x})]\,(\boldsymbol{\omega}) = \mathcal{F}_x[B(\mathbf{x})f(\mathbf{x})]\,(\boldsymbol{\omega})$$

2 DERIVATION FOR THE BILOBE PHASOR SINEWAVE

The paper introduces phasor noise and phasor sinewave using sine waves and the phasor addition for the bilobe case. In this supplemental we rely on complex notations (see section 3.1 of the paper) from the start to simplify (and homogenize) the derivations.

2.1 Complex notation

Lets define the bilobe complex Gabor noise as :

$$\begin{aligned}\mathcal{G}_{\mathbf{u}}(\mathbf{x}) &= \sum_j a_j(\mathbf{x}) e^{i(\mathbf{x}-\mathbf{x}_j)\cdot\mathbf{u}} \\ &= e^{i\mathbf{x}\mathbf{u}} \sum_j a_j(\mathbf{x}) e^{-i\mathbf{x}_j\cdot\mathbf{u}} \\ &= e^{i\mathbf{x}\mathbf{u}} \sum_j a_j(\mathbf{x}) e^{i\varphi_j}\end{aligned}$$

where $\|\mathbf{u}\|$ defines the main frequency of the noise, $\mathbf{u}/\|\mathbf{u}\|$ defines the direction of anisotropy and $\varphi_j = -\mathbf{x}_j \cdot \mathbf{u}$. Contrary to the paper's notations, here we include the main frequency F in the direction of anisotropy $\mathbf{u}$ for the sake of following derivations' clarity.

The previous factorisation allows to reveal the phasor field presented in section 3.1.1 of the paper. It also provides an alternative

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way to derive formulas obtained from the phasor addition (equation (6) and (7) of the paper). Indeed, we have :

$$\sum_j a_j(\mathbf{x})e^{i\varphi_j} = I(\mathbf{x})e^{i\varphi(\mathbf{x})}$$

where the phase field φ is defined as:

$$\varphi(\mathbf{x}) = \mathrm{Arg}\,(\sum_j a_j(\mathbf{x})e^{i\varphi_j})$$

and the intensity field I is defined as:

$$I(\mathbf{x}) = \| \sum_j a_j(\mathbf{x})e^{i\varphi_j} \|$$

The Phasor noise is defined by :

$$\phi(\mathbf{x}) = \mathrm{Arg}\,(\mathcal{G}_{\mathbf{u}}(\mathbf{x})) = \mathbf{x}\cdot\mathbf{u} + \varphi(\mathbf{x})$$

and the Phasor Sinewave P can be defined by the four following equivalent definitions :

$$\begin{aligned} P(x) &= \sin(\phi(\mathbf{x})) \\ &= \frac{\mathrm{Im}(\mathcal{G}_{\mathbf{u}}(\mathbf{x}))}{\|\mathcal{G}_{\mathbf{u}}(\mathbf{x})\|} \\ &= \mathrm{Im}(e^{i\phi(\mathbf{x})}) \\ &= \frac{1}{2i}(e^{i\phi(\mathbf{x})} - e^{-i\phi(\mathbf{x})}) \end{aligned}$$

Using the last formula, we obtain:

$$\begin{aligned} P(\mathbf{x}) &= \frac{1}{2i}\left(e^{i(\mathbf{x}\cdot\mathbf{u}+\varphi(\mathbf{x}))} - e^{-i(\mathbf{x}\cdot\mathbf{u}+\varphi(\mathbf{x}))}\right) \\ &= \frac{1}{2i}e^{i\mathbf{x}\cdot\mathbf{u}}e^{i\varphi(\mathbf{x})} - \frac{1}{2i}e^{-i\mathbf{x}\cdot\mathbf{u}}e^{-i\varphi(\mathbf{x})} \end{aligned}$$

2.2 Spectrum of P

In this section, we derive formulas of section 3.1.2 of the paper. We now look at the Fourier transform of $B(\mathbf{x})P(\mathbf{x})$ to verify that the main frequency of the original noise are preserved. The spectrum of $B(\mathbf{x})P(\mathbf{x})$ is, by linearity of the Fourier transform :

$$\mathcal{F}_B[P(\mathbf{x})]\,(\boldsymbol{\omega}) = \frac{1}{2i}\mathcal{F}_B[e^{i\mathbf{x}\cdot\mathbf{u}}e^{i\varphi(\mathbf{x})}]\,(\boldsymbol{\omega}) - \frac{1}{2i}\mathcal{F}_B[e^{-i\mathbf{x}\cdot\mathbf{u}}e^{-i\varphi(\mathbf{x})}]\,(\boldsymbol{\omega})$$

By applying the convolution theorem to the first term, we obtain :

$$\begin{aligned} \mathcal{F}_B[e^{i\mathbf{x}\cdot\mathbf{u}}e^{i\varphi(\mathbf{x}))}]\,(\boldsymbol{\omega}) &= \left(\mathcal{F}_x[e^{i\mathbf{x}\cdot\mathbf{u}}] \circledast \mathcal{F}_B[e^{i\varphi(\mathbf{x})}]\right)(\boldsymbol{\omega}) \\ &= \left(\delta\left(. - \frac{\mathbf{u}}{2\pi}\right) \circledast \mathcal{F}_B[e^{i\varphi(\mathbf{x})}]\right)(\boldsymbol{\omega}) \\ &= \mathcal{F}_B[e^{i\varphi(\mathbf{x})}]\left(\boldsymbol{\omega} - \frac{\mathbf{u}}{2\pi}\right) \end{aligned}$$

By applying similar derivation to the second term, we obtain :

$$\begin{aligned} \mathcal{F}_B[P(\mathbf{x})]\,(\boldsymbol{\omega}) =& \frac{1}{2i}\mathcal{F}_B[e^{i\varphi(\mathbf{x})}]\left(\boldsymbol{\omega} - \frac{\mathbf{u}}{2\pi}\right) \\ &- \frac{1}{2i}\mathcal{F}_B[e^{-i\varphi(\mathbf{x})}]\left(\boldsymbol{\omega} + \frac{\mathbf{u}}{2\pi}\right) \end{aligned}$$

Therefore the main frequencies are preserved as the two lobes are centered on the same main frequencies as the original Gabor noise.

Lets take a look at one of the two term of the Fourier transform of the Phasor Sinewave, we have:

$$\mathcal{F}_B[e^{i\varphi(\mathbf{x})}] = \mathcal{F}_B[\cos(\varphi(\mathbf{x}))] + i\mathcal{F}_B[\sin(\varphi(\mathbf{x}))]$$

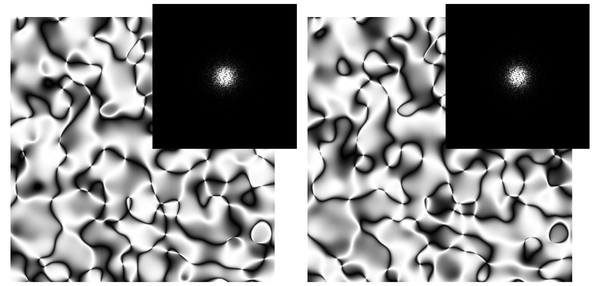

Fig. 1. Visualizing $\mathrm{abs}\,(\sin\varphi((\mathbf{x})))$ and $\mathrm{abs}\,(\cos\varphi((\mathbf{x})))$ as well as their Fourier transforms. Both have rapidly decreasing frequency content. The singularities are visible in the fields.

and

$$\sin(\varphi(\mathbf{x})) = \frac{\sum_j a_j(\mathbf{x})\sin\varphi_j}{I(\mathbf{x})} \quad \text{and} \quad \cos(\varphi(\mathbf{x})) = \frac{\sum_j a_j(\mathbf{x})\cos\varphi_j}{I(\mathbf{x})}$$

noting $I(\mathbf{x}) = \| \sum_j a_j(\mathbf{x})e^{i\varphi_j} \|$ which is also equal to $\|\mathcal{G}_{\mathbf{u}}(\mathbf{x})\|$. Previous equalities can be rewritten as :

$$\sin(\varphi(\mathbf{x})) = \frac{\mathrm{sign}\,(\mathcal{S}(\mathbf{x}))}{\sqrt{1+\mathcal{T}^{-2}(\mathbf{x})}} \quad \text{and} \quad \cos(\varphi(\mathbf{x})) = \frac{\mathrm{sign}\,(C(\mathbf{x}))}{\sqrt{1+\mathcal{T}^{2}(\mathbf{x})}}$$

with

$$\mathcal{S}(\mathbf{x}) = \sum_j a_j(\mathbf{x})\sin\varphi_j$$

$$C(\mathbf{x}) = \sum_j a_j(\mathbf{x})\cos\varphi_j$$

$$\mathcal{T}(\mathbf{x}) = \frac{\sum_j a_j(\mathbf{x})\sin\varphi_j}{\sum_j a_j(\mathbf{x})\cos\varphi_j}$$

Finally, we note that in the one dimensional case, we have :

$$\mathcal{F}_B[\cos(\varphi(\mathbf{x}))](\boldsymbol{\omega}) = \mathcal{F}_x[\frac{B(\mathbf{x})\,\mathrm{sign}\,(C(\mathbf{x}))}{\sqrt{1+\mathcal{T}^2(\mathbf{x})}}](\boldsymbol{\omega})$$

which is the Fourier transform of an integrable smooth function. Indeed, the denominator $1+\mathcal{T}^2$ is in the range $[1, +\infty[$, and the discontinuities of $\mathrm{sign}\,(C(\mathbf{x}))$ correspond to the cases where $B(\mathbf{x})/(1+\mathcal{T}^2(\mathbf{x}))$ tend toward 0. This is true for any degree of B-spline, therefore we can choose B such that the product is both of class C^1 and that $f^{(1)}$ is bounded, hence we have :

$$\exists M \text{ such that } \quad \mathcal{F}_B[\cos(\varphi(x))](\omega) \leq M\omega^{-1}$$

thanks to the relationship between smoothness and decay of the Fourier transform. Therefore, when ω tends towards $\pm\infty$, the function $\mathcal{F}_B[\cos(\varphi(x))](\omega)$ is rapidly decreasing.

In 2D and 3D, $\cos(\varphi(x))$ and $\sin(\varphi(x))$ are smooth everywhere except in a few singular points — this is where the phasor field exhibits singularities. There, only directional derivatives are defined. However, we can verify numerically that the shape of the Fourier transform remains a lobe, see Figure 1.

2.3 Spectrum of variance of P

In this section, we derive formulas of section 3.1.3 of the paper. We will now take a look at the spectrum of variance of the Phasor Sinewave (i.e. the spectrum of P^2) in order to show the absence of contrast fluctuations (no central lobe).

We use the same approach as for the spectrum of the noise itself:

$$P^2(\mathbf{x}) = \sum_{k=-\infty}^{+\infty} B(x + kT_0)P^2(x + kT_0) = \left(\left(\sum_{k=-\infty}^{+\infty} \delta_{kT_0}\right) \circledast B\, P^2\right)(\mathbf{x})$$

Lets first take a look at the development of the formula of P^2 :

$$\begin{aligned} P^2(\mathbf{x}) &= -\frac{1}{4}\left(e^{i(\mathbf{x}\cdot\mathbf{u}+\varphi(\mathbf{x})} - e^{-i(\mathbf{x}\cdot\mathbf{u}+\varphi(\mathbf{x}))}\right)^2 \\ &= \frac{1}{2} - \frac{1}{4}e^{2i\mathbf{x}\cdot\mathbf{u}}e^{2i\varphi(\mathbf{x})} - \frac{1}{4}e^{-2i\mathbf{x}\cdot\mathbf{u}}e^{-2i\varphi(\mathbf{x})} \end{aligned}$$

Noting that the two non-constant terms are the same as for P with exponential arguments varying twice as rapidly, we obtain :

$$\begin{aligned} \mathcal{F}_B[P^2(\mathbf{x})]((\boldsymbol{\omega})) =& \frac{1}{2}\delta_0(\boldsymbol{\omega}) \\ &- \frac{1}{4}\mathcal{F}_B[e^{2i\varphi(\mathbf{x})}]\left(\boldsymbol{\omega} - \frac{\mathbf{u}}{\pi}\right) \\ &- \frac{1}{4}\mathcal{F}_B[e^{-2i\varphi(\mathbf{x})}]\left(\boldsymbol{\omega} + \frac{\mathbf{u}}{\pi}\right) \end{aligned}$$

Therefore we can observe the absence of a central lobe and the presence of two lobes at twice the frequencies of the original noise. Note that the central lobe is replaced by a Dirac corresponding to the average value of the variance (which is non-zero).

Finally, we can note that the suppression of the *main* source of low frequency in the spectrum of variance (eg the removal of the local loss of contrast) comes from the fact that the real signal/noise is defined as the substraction (or sum if we take the real part) of two complex exponential which are conjugate of one another multiplied by a *constant* value. This sufficient condition opens alternative directions to define the phase function ϕ.

3 TWO BILOBE PHASOR SINWAVE

In this section, we provide an alternative point of view on discussion of section 3.2.1 of the paper. We will now study the Phasor sinwave defined from the interactions between two complex bilobe Gabor noises $\mathcal{G}_{\mathbf{u}_k}$ and $\mathcal{G}_{\mathbf{u}_m}$. In this case the Phasor noise is defined as :

$$\phi(\mathbf{x}) = \mathrm{Arg}\left(\mathcal{G}_s(\mathbf{x})\right)$$

with

$$\mathcal{G}_s(\mathbf{x}) = (\mathcal{G}_{\mathbf{u}_k} + \mathcal{G}_{\mathbf{u}_m})(\mathbf{x})$$

The Phasor Sinwave is defined as :

$$\begin{aligned} P(\mathbf{x}) &= \mathrm{Im}(e^{i\phi(\mathbf{x})}) \\ &= \frac{1}{2i}(e^{i\phi(\mathbf{x})} - e^{-i\phi(\mathbf{x})}) \\ &= \frac{\mathrm{Im}(\mathcal{G}_s(\mathbf{x}))}{|\mathcal{G}_s(\mathbf{x})|} \end{aligned}$$

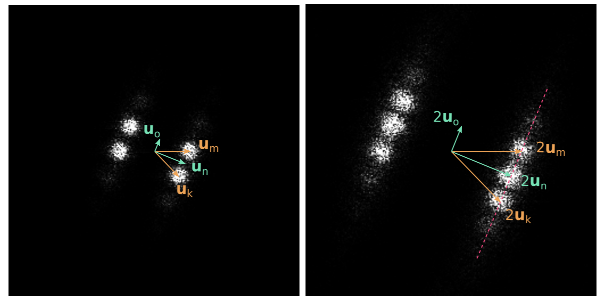

Fig. 2. From left to right, spectrum and spectrum of variance of a phasor sinewave defined by the summation of two complex Gabor noise. Frequency vectors $\mathbf{u}_k$ and $\mathbf{u}_m$ of the two Gabor noises are displayed in orange. Alternative basis $(\mathbf{u}_n, \mathbf{u}_o)$ is displayed in teal.

3.1 Alternative basis

We will first express $\mathcal{G}_s$ in the basis defined by :

$$\mathbf{u}_n = \frac{1}{2}(\mathbf{u}_m + \mathbf{u}_k)$$

and

$$\mathbf{u}_o = \frac{1}{2}(\mathbf{u}_m - \mathbf{u}_k)$$

and shown in Figure 2. If we substract $\mathbf{u}_n$ respectively to $\mathbf{u}_k$ and $\mathbf{u}_m$ we obtain:

$$\mathbf{u}_k - \mathbf{u}_n = \frac{1}{2}(\mathbf{u}_k - \mathbf{u}_m) = \mathbf{u}_o$$

and

$$\mathbf{u}_m - \mathbf{u}_n = \frac{1}{2}(\mathbf{u}_m - \mathbf{u}_k) = -\mathbf{u}_o$$

This property allows to perform the following factorisation :

$$\begin{aligned} \mathcal{G}_s(\mathbf{x}) &= \sum_j a_j(\mathbf{x})e^{i(\mathbf{x}-\mathbf{x}_j)\cdot\mathbf{u}_k} + \sum_l a_l(\mathbf{x})e^{i(\mathbf{x}-\mathbf{x}_l)\cdot\mathbf{u}_m} \\ &= e^{i\mathbf{x}\cdot\mathbf{u}_n}\left(\sum_j a_j(\mathbf{x})e^{i(-\mathbf{x}\cdot\mathbf{u}_o-\mathbf{x}_j\cdot\mathbf{u}_k)} + \sum_l a_l(\mathbf{x})e^{i(\mathbf{x}\cdot\mathbf{u}_o-\mathbf{x}_l\cdot\mathbf{u}_m)}\right) \\ &= e^{i\mathbf{x}\cdot\mathbf{u}_n}\left(\sum_j a_j(\mathbf{x})e^{i(-\mathbf{x}\cdot\mathbf{u}_o-\tilde{\varphi}_j)} + \sum_l a_l(\mathbf{x})e^{i(\mathbf{x}\cdot\mathbf{u}_o-\tilde{\varphi}_l)}\right) \\ &=: e^{i\mathbf{x}\cdot\mathbf{u}_n}\Sigma_o(\mathbf{x}) \end{aligned}$$

where a unit complex number with linearly varying argument in the direction $\mathbf{u}_n$ is modulated by a complex noise Σ_o of main direction u_o. Note that contrary to the bilobe case, we still have complex exponential with argument function of $\mathbf{x}$ that cannot be complately factorised ($e^{-i\mathbf{x}\cdot\mathbf{u}_o}$ and $e^{i\mathbf{x}\cdot\mathbf{u}_o}$).

3.2 Spectrum of variance

First, lets verify the integrability of the periodized noise P^2 over a period. We have :

$$\begin{aligned}
P^2(\mathbf{x}) &= \frac{1}{\mathcal{G}_s(\mathbf{x})\overline{\mathcal{G}_s}(\mathbf{x})}\left(\frac{1}{2i}(\mathcal{G}_s(\mathbf{x}) - \overline{\mathcal{G}_s}(\mathbf{x}))\right)^2 \\
&= \frac{1}{4}\frac{1}{\mathcal{G}_s(\mathbf{x})\overline{\mathcal{G}_s}(\mathbf{x})}\left(\mathcal{G}_s^2(\mathbf{x}) - 2\mathcal{G}_s(\mathbf{x})\overline{\mathcal{G}_s}(\mathbf{x}) + \overline{\mathcal{G}_s}^2(\mathbf{x})\right) \\
&= \frac{1}{2} - \frac{1}{4}\frac{\mathcal{G}_s^2(\mathbf{x}) + \overline{\mathcal{G}_s}^2(\mathbf{x})}{\mathcal{G}_s(\mathbf{x})\overline{\mathcal{G}_s}(\mathbf{x})} \\
&= \frac{1}{2} - \frac{1}{4}\left(\frac{\mathcal{G}_s(\mathbf{x})}{\overline{\mathcal{G}_s}(\mathbf{x})} + \frac{\overline{\mathcal{G}_s}(\mathbf{x})}{\mathcal{G}_s(\mathbf{x})}\right) \\
&= \frac{1}{2}\left(1 - \mathrm{Re}(\frac{\mathcal{G}_s(\mathbf{x})}{\overline{\mathcal{G}_s}(\mathbf{x})})\right)
\end{aligned}$$

and we have

$$\begin{aligned}
\mathrm{Re}(\frac{\mathcal{G}_s}{\overline{\mathcal{G}_s}}) &= \frac{\mathrm{Re}^2(\mathcal{G}_s) - \mathrm{Im}^2(\mathcal{G}_s)}{\mathrm{Re}^2(\mathcal{G}_s) + \mathrm{Im}^2(\mathcal{G}_s)} \\
&= \frac{1}{1 + \frac{\mathrm{Im}^2(\mathcal{G}_s)}{\mathrm{Re}^2(\mathcal{G}_s)}} - \frac{1}{1 + \frac{\mathrm{Re}^2(\mathcal{G}_s)}{\mathrm{Im}^2(\mathcal{G}_s)}}
\end{aligned}$$

which is bounded as the denominators are contained in the interval $[1, +\infty]$, the Fourier transformed of the periodized version of the squared phasor noise is well defined since $B\,P^2$ is integrable.

From previous derivations, we also have :

$$P^2(\mathbf{x}) = \frac{1}{2} - \frac{1}{4}\frac{\mathcal{G}_s^2(\mathbf{x}) + \overline{\mathcal{G}_s}^2(\mathbf{x})}{\mathcal{G}_s(\mathbf{x})\overline{\mathcal{G}_s}(\mathbf{x})}$$

and

$$\mathcal{G}_s(\mathbf{x})\overline{\mathcal{G}_s}(\mathbf{x}) = \mathrm{Re}^2(\Sigma_o(\mathbf{x})) + \mathrm{Im}^2(\Sigma_o(\mathbf{x})) = |\Sigma_o(\mathbf{x})|^2$$

$$\mathcal{G}_s^2(\mathbf{x}) = e^{i2\mathbf{x}\cdot\mathbf{u}_n}(\Sigma_o(\mathbf{x}))^2$$

$$\overline{\mathcal{G}_s}^2(\mathbf{x}) = e^{-i2\mathbf{x}\cdot\mathbf{u}_n}(\overline{\Sigma_o(\mathbf{x})})^2$$

then

$$\begin{aligned}
P^2(\mathbf{x}) &= \frac{1}{2} - \frac{1}{4}\frac{e^{i2\mathbf{x}\cdot\mathbf{u}_n}(\Sigma_o(\mathbf{x}))^2 + e^{-i2\mathbf{x}\cdot\mathbf{u}_n}(\overline{\Sigma_o(\mathbf{x})})^2}{|\Sigma_o(\mathbf{x})|^2} \\
&= \frac{1}{2} - \frac{1}{4}(e^{i2\mathbf{x}\cdot\mathbf{u}_n}\frac{(\Sigma_o(\mathbf{x}))^2}{|\Sigma_o(\mathbf{x})|^2} + e^{-i2\mathbf{x}\cdot\mathbf{u}_n}\frac{(\overline{\Sigma_o(\mathbf{x})})^2}{|\Sigma_o(\mathbf{x})|^2})
\end{aligned}$$

Applying the Convolution theorem to $\mathcal{F}_B[P^2(\mathbf{x})]$, we obtain :

$$\begin{aligned}
\mathcal{F}_B[P^2(\mathbf{x})]\,(\boldsymbol{\omega}) =& \frac{1}{2}\delta_0\,(\boldsymbol{\omega}) \\
&+ \frac{1}{4}\delta_{\frac{\mathbf{u}_n}{\pi}} \circledast \mathcal{F}_B[\frac{(\Sigma_o(\mathbf{x}))^2}{|\Sigma_o(\mathbf{x})|^2}]\,(\boldsymbol{\omega}) \\
&+ \frac{1}{4}\delta_{-\frac{\mathbf{u}_n}{\pi}} \circledast \mathcal{F}_B[\frac{(\overline{\Sigma_o(\mathbf{x})})^2}{|\Sigma_o(\mathbf{x})|^2}]\,(\boldsymbol{\omega})
\end{aligned}$$

or equivalently :

$$\begin{aligned}
\mathcal{F}_B[P^2(\mathbf{x})]\,(\boldsymbol{\omega}) =& \frac{1}{2}\delta_0\,(\boldsymbol{\omega}) \\
&+ \frac{1}{4}\mathcal{F}_B[\frac{(\Sigma_o(\mathbf{x}))^2}{|\Sigma_o(\mathbf{x})|^2}]\left(\boldsymbol{\omega} + \frac{\mathbf{u}}{\pi}\right) \\
&+ \frac{1}{4}\mathcal{F}_B[\frac{(\overline{\Sigma_o(\mathbf{x})})^2}{|\Sigma_o(\mathbf{x})|^2}]\left(\boldsymbol{\omega} - \frac{\mathbf{u}}{\pi}\right)
\end{aligned}$$

First, $\frac{(\Sigma_o(\mathbf{x}))^2}{|\Sigma_o(\mathbf{x})|^2}$ is a noise with anistropy along the direction $\mathbf{u}_o$, hence its Fourier transform is stretched in this direction. Secondly, formula for $\mathcal{F}_B[P^2(\mathbf{x})]\,(\boldsymbol{\omega})$ shows that two such shapes are translated by $\frac{1}{\pi}\mathbf{u}_n$ in the frequency domain, hence if the line passing by $\frac{1}{\pi}\mathbf{u}_n$ with direction $\mathbf{u}_o$ does not come too close (relatively to the bandwith of the Gabor noises) to the origin of the frequency domain then no low frequency are present in the spectrum of variance (see Figure 2).

4 INVARIANCE TO THE DEGREE OF B-SPLINE USED IN THE ANALYSIS

We now demonstrate that the choice of the B-spline degree used for windowing does not have any impact on our Fourier transform. The univariate cardinal B-spline are defined by:

$$B_0 : x \longrightarrow \mathrm{rect}(\frac{x}{T_0})$$

$$B_n = \frac{1}{T_0}B_0 \circledast B_{n-1}$$

with T_0 the period of the noise (see section 1). Their Fourier transforms are:

$$\begin{aligned}
\mathcal{F}_x[B_0](\omega) &= T_0\mathcal{F}_x[\mathrm{rect}(x)](T_0\omega) \\
&= T_0\,\mathrm{sinc}(T_0\omega)
\end{aligned}$$

and

$$\mathcal{F}_x[B_n](\omega) = T_0\,\mathrm{sinc}^n(T_0\omega).$$

Lets define an arbitrary periodic function by its Fourier serie:

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k e^{2i\pi kxT_0}$$

Then, we have :

$$\begin{aligned}
\mathcal{F}_x[B_n(x)f(x)](\omega) &= \left(\mathcal{F}_x[B_n(x)] \circledast \mathcal{F}_x[\sum_{k=-\infty}^{+\infty} c_k e^{2i\pi kxT_0}]\right)(\omega) \\
&= \left(\mathcal{F}_x[B_n(x)] \circledast \sum_{k=-\infty}^{+\infty} c_k\delta(\omega - k\omega_0)\right)(\omega) \\
&= \sum_{k=-\infty}^{+\infty} c_k\mathcal{F}_x[B_n(x)](\omega - k\omega_0) \\
&= \sum_{k=-\infty}^{+\infty} c_kT_0\,\mathrm{sinc}^n(T_0(\omega - k\omega_0))
\end{aligned}$$

hence, we obtain the coefficient of the Fourier transform of f :

$$\mathcal{F}_x[B^n(x)f(x)](k\omega_0) = c_k = \mathcal{F}_x[f(x)](k\omega_0)$$